TIT/HEP-572
HIP-2007-29/TH
May, 2007

Deformation of Super Yang-Mills Theories in R-R 3-form Background

Katsushi Ito[1], Hiroaki Nakajima[2] and Shin Sasaki[3]

[1] *Department of Physics*
Tokyo Institute of Technology
Tokyo, 152-8551, Japan

[2] *Department of Physics and Institute of Basic Science*
Sungkyunkwan University
Suwon, 440-746, Korea

[3] *High Energy Physics Division, Department of Physical Sciences*
Helsinki Institute of Physics
University of Helsinki
P.O.Box 64, FIN-00014, Finland

Abstract

We study deformation of $\mathcal{N}=2$ and $\mathcal{N}=4$ super Yang-Mills theories, which are obtained as the low-energy effective theories on the (fractional) D3-branes in the presence of constant Ramond-Ramond 3-form background. We calculate the Lagrangian at the second order in the deformation parameter from open string disk amplitudes. In $\mathcal{N}=4$ case we find that all supersymmetries are broken for generic deformation parameter but part of supersymmetries are unbroken for special case. We also find that classical vacua admit fuzzy sphere configuration. In $\mathcal{N}=2$ case we determine the deformed supersymmetries. We rewrite the deformed Lagrangians in terms of $\mathcal{N}=1$ superspace, where the deformation is interpreted as that of coupling constants.

Contents

1 Introduction

Low-energy effective field theories on D-branes in closed string backgrounds have attracted much attentions. The effects of the Ramond-Ramond (R-R) backgrounds are particularly

interesting for studying (non-)perturbative properties of supersymmetric gauge theories and superstrings. For example, the constant graviphoton background, which comes from the self-dual R-R 5-form field strenth wrapping three cycle in a Calabi-Yau manifold, produces stringy corrections to the F-terms in effective theories [1, 2]. Such corrections play an important role in studying non-perturbative properties of supersymmetric gauge theories [3, 4]. Closed string background is also interesting from the geometrical point of view because it deforms the world-volume geometry of D-branes. A well-known example is the constant NS-NS B-field, which leads to the noncommutative space-time realized by the Moyal product [5, 6]. The R-R background also deforms the world-volume geometry. In fact, the constant self-dual R-R 5-form background on the fractional D3-branes introduces non(anti)commutativity of (Euclidean) superspace [7, 8, 9]. The deformed supersymmetric gauge theories on non(anti)commutative superspace are studied extensively [10, 11, 12, 13, 14].

Since superstring theory contains R-R fields, it would be interesting to study deformed supersymmetric gauge theories in R-R backgrounds with various ranks and their (non-)perturbative properties. Recently Billó *et al* [15] studied the effective action on the fractional D3-D(-1) system in the R-R 3-form background $\mathcal{F}$ with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$ in the zero slope limit. They showed that the deformed action agrees with the instanton effective action of $\mathcal{N}=2$ supersymmetric gauge theory in the Ω-background [3] at the lowest order of the deformation parameter and gauge coupling constant. The Ω-background utilize the integral over the instanton moduli space [3, 16]. This type of deformation is not obtained from the non(anti)commutative deformation of superspace. It is an interesting problem to study geometrical meaning of this deformation.

In order to examine the effects of R-R background, the most direct approach is to calculate the low-energy effective action on the D-branes from superstring theory. One can compute the action of non(anti)commutative gauge theories directly from the effective action on the (fractional) D3-branes [17, 15, 18, 19], where interaction terms are obtained from the open string disk amplitudes with insertion of graviphoton vertex operators. For example, the deformed action of $\mathcal{N}=1$ supersymmetric gauge theories was derived from the fractional D3-branes in type IIB superstring theories compactified on $\mathbf{C}^3/\mathbf{Z}_2\times\mathbf{Z}_2$ [17]. The effective theory is $\mathcal{N}=1$ super Yang-Mills theory on non(anti)commutative $\mathcal{N}=1$

superspace [10].

In [18, 19] we discussed the deformation of $\mathcal{N}=2$ and $\mathcal{N}=4$ super Yang-Mills theories in the R-R background field strength of the form $\mathcal{F}^{\alpha\beta AB}$, where α and β label the spinor indices of (Euclidean) space-time and A and B are internal spinor indices. We classify the field strength into four types $\mathcal{F}^{(\alpha\beta)(AB)}$, $\mathcal{F}^{[\alpha\beta](AB)}$, $\mathcal{F}^{(\alpha\beta)[AB]}$ and $\mathcal{F}^{[\alpha\beta][AB]}$. Here (ab) ($[ab]$) denotes the (anti)symmetrization of ab. We call these deformations as (S,S), (A,S), (S,A) and (A,A)-type, respectively, where the (S,S)-type deformation with fixed $(2\pi\alpha')^{3/2}\mathcal{F}$ corresponds to the case studied in [17]. In [18], we studied the first order correction to $\mathcal{N}=2$ super Yang-Mills action from the (S,S)-type background with fixed $(2\pi\alpha')^{3/2}\mathcal{F}$. We showed that deformed theory agrees with $\mathcal{N}=2$ super Yang-Mills theory on non(anti)commutative $\mathcal{N}=2$ harmonic superspace [11, 12, 13]. In [19], we studied the first order correction to $\mathcal{N}=4$ super Yang-Mills theory in (S,S)-type background with fixed $(2\pi\alpha')^{3/2}\mathcal{F}$. By restricting the deformation parameter to the special case, the deformed Lagrangian is reduced to the one in non(anti)commutative $\mathcal{N}=1$ superspace. Therefore it is natural to think that the (S,S)-type deformation with fixed $(2\pi\alpha')^{3/2}\mathcal{F}$ corresponds to the non(anti)commutative deformation of $\mathcal{N}(\leq 4)$ extended superspace at full order in deformation parameter. On the other hand, the index structure of the (A,A) type background suggests that it corresponds to the singlet deformation of extended superspace [12, 20], although we need to take into account the backreaction to the closed string backgrounds [18]. The (S,A) and (A,S) type deformations with fixed $(2\pi\alpha')^{3/2}\mathcal{F}$ would also provide nontrivial deformation of supersymmetric gauge theories, which cannot be realized as non(anti)commutative superspace. However, it is difficult to compute the deformed actions due to its complicated structure.

As shown in [15], the (S,A)-type background with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$ provides nontrivial deformation of $\mathcal{N}=2$ super Yang-Mills theory, which is useful for studying instanton calculus. Hence it would be an interesting problem to work out the deformations by the constant R-R backgrounds with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$ and their non-perturbative properties. The purpose of this paper is to study the deformation of super Yang-Mills theories with $\mathcal{N}=2$ and 4 supersymmetries corresponding to the (S,A) and (A,S)-types background with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$.

We will calculate disk amplitudes with one R-R vertex operator and derive the effective

action on the (fractional) D-branes. For $\mathcal{N} = 4$ case, we will show that the bosonic action agrees with the Chern-Simons action with the (dual) R-R potentials [21]. The deformed scalar potential has nontrivial minima. Actually, for both (S,A) and (A,S)-type deformations of $\mathcal{N} = 4$ super Yang-Mills theory, we find a fuzzy sphere configuration [22, 19] for adjoint scalars. In general number of unbroken supersymmetries are restricted on the D-branes in the presence of R-R backgrounds. We will examine invariance of the deformed Lagrangian under remaining supersymmetries. The deformation of $\mathcal{N} = 2$ super Yang-Mills theory is obtained from $\mathcal{N} = 4$ theory by the reduction due to the $\mathbf{Z}_2$ orbifold of $\mathbf{C}^2$. For both $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases, we are able to explore geometrical interpretation of this deformation in terms of superspace formalism. We will show that (S,A) and (A,S)-type deformations with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$ are realized by introducing superspace dependent coupling constants. This is in contrast with the case with the (S,S)-type deformation with fixed $(2\pi\alpha')^{3/2}\mathcal{F}$, where its deformation is realized by the star product for supercoordinates.

This paper is organized as follows: In section 2, we calculate the (S,A) and (A,S)-type background corrections to $\mathcal{N} = 4$ super Yang-Mills theory from the open string disk amplitudes with one closed string R-R vertex operator. Unbroken supersymmetries are classified in terms of the rank of deformation parameter in some cases. The fuzzy sphere configurations of vacuum in the deformed theories are investigated. In section 3, we confirm that the R-R correction terms in (S,A) and (A,S)-type deformed $\mathcal{N} = 4$ theories are consistent with the Chern-Simons term of the D-brane effective action coupled to the R-R potential. In section 4, we study the (S,A) and (A,S)-type deformations of $\mathcal{N} = 2$ super Yang-Mills theory and its deformed supersymmetry. In section 5 we rewrite the deformed action in terms of $\mathcal{N} = 1$ superspace and show that (A,S)-type deformation is regarded as the mass deformation of super Yang-Mills theory. Section 6 is devoted to conclusions and discussion.

2 Deformed $\mathcal{N}=4$ Super Yang-Mills theory in R-R 3-form background

In this section we study the low-energy effective action on D3-branes in type IIB superstrings from the disk amplitudes with one R-R vertex operator of (S,A) or (A,S)-type. Here we use NSR formalism and introduce spin fields [23, 24] to represent space-time spinor. The low-energy effective field theory on N D3-branes are described by gauge fields A_μ ($\mu=1,2,3,4$), six real scalars φ^a ($a=5,\ldots,10$) and Weyl fermions $\Lambda_\alpha{}^A$ and $\bar{\Lambda}^{\dot{\alpha}}{}_A$ ($A=1,2,3,4$), which belong to the adjoint representation of gauge group $U(N)$. We denote T^m as the basis of $U(N)$ generators normalized as $\mathrm{Tr}(T^mT^n)=k\delta^{mn}$ with constant factor k.

The vertex operators for these fields are [26]

$$
\begin{aligned}
V_A^{(-1)}(y;p) &= (2\pi\alpha')^{\frac{1}{2}}\frac{A_\mu(p)}{\sqrt{2}}\psi^\mu(y)e^{-\phi(y)}e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}, \\
V_A^{(0)}(y;p) &= 2i(2\pi\alpha')^{\frac{1}{2}}A_\mu(p)\left(\partial X^\mu(y)+i(2\pi\alpha')^{\frac{1}{2}}p\cdot\psi\psi^\mu(y)\right)e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}. \quad (2.1)
\end{aligned}
$$

$$
\begin{aligned}
V_\varphi^{(-1)}(y;p) &= (2\pi\alpha')^{\frac{1}{2}}\frac{\varphi_a(p)}{\sqrt{2}}\psi^a(y)e^{-\phi(y)}e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}, \\
V_\varphi^{(0)}(y;p) &= 2i(2\pi\alpha')^{\frac{1}{2}}\varphi_a(p)\left(\partial X^a(y)+i(2\pi\alpha')^{\frac{1}{2}}p\cdot\psi\psi^a(y)\right)e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}. \quad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
V_\Lambda^{(-1/2)}(y;p) &= (2\pi\alpha')^{\frac{3}{4}}\Lambda^{\alpha A}(p)S_\alpha(y)S_A(y)e^{-\frac{1}{2}\phi(y)}e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}, \\
V_{\bar{\Lambda}}^{(-1/2)}(y;p) &= (2\pi\alpha')^{\frac{3}{4}}\bar{\Lambda}_{\dot{\alpha}A}(p)S^{\dot{\alpha}}(y)S^A(y)e^{-\frac{1}{2}\phi(y)}e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}. \quad (2.3)
\end{aligned}
$$

Here $(X^M(z),\psi^M(z))$ ($M=1,\ldots,10$) are free bosons and fermions on the worldsheet, where μ labels the worldvolume coordinates on D3-branes and a coordinates transverse to the worldvolume of the D3-branes. S_α and S_A denote the spin operators for space-time and internal space parts. ϕ is a free boson obtained from the bosonization of the bosonic ghost (β,γ). For gauge fields and scalar fields we use two physically equivalent vertex operators with picture number -1 and 0. For fermions we use the vertex operator with picture number $-1/2$.

The disk amplitudes in the zero-slope limit $\alpha'\to 0$ reproduce the action of $\mathcal{N}=4$ super Yang-Mills theory. It is convenient to introduce auxiliary field vertex operators in

order to reduce higher point amplitudes to the lower ones [25, 26, 18, 19]. These are given by

$$
\begin{aligned}
V^{(0)}_{H_{AA}}(y;p) &= \frac{1}{2}(2\pi\alpha')H_{\mu\nu}(p)\psi^\mu\psi^\nu(y)e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}, \\
V^{(0)}_{H_{A\varphi}}(y;p) &= 2(2\pi\alpha')H_{\mu a}(p)\psi^\mu\psi^a(y)e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}, \\
V^{(0)}_{H_{\varphi\varphi}}(y;p) &= -\frac{1}{\sqrt{2}}(2\pi\alpha')H_{ab}(p)\psi^a\psi^b(y)e^{i\sqrt{2\pi\alpha'}p\cdot X(y)}.
\end{aligned}
\tag{2.4}
$$

Note that these vertex operators are not BRST invariant. The total Lagrangian includes only the cubic interaction terms and becomes

$$
\begin{aligned}
\mathcal{L}_{\mathcal{N}=4} \;=\; & -\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\frac{1}{2}(\partial_\mu A_\nu-\partial_\nu A_\mu)\partial^\mu A^\nu + i\partial_\mu A_\nu[A^\mu,A^\nu]+\frac{1}{2}H_cH^c+\frac{1}{2}H_c\eta^c_{\mu\nu}[A^\mu,A^\nu]\right] \\
& -\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\frac{1}{2}H_{ab}H_{ab}+\frac{1}{\sqrt{2}}H_{ab}[\varphi_a,\varphi_b]\right] \\
& -\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a + i\partial_\mu\varphi_a[A^\mu,\varphi_a]+\frac{1}{2}H_{\mu a}H^{\mu a}+H_{\mu a}[A^\mu,\varphi_a]\right] \\
& -\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[i\Lambda^A\sigma^\mu D_\mu\bar{\Lambda}_A-\frac{1}{2}\left(\Sigma^a\right)^{AB}\bar{\Lambda}_{\dot\alpha A}[\varphi_a,\bar{\Lambda}^{\dot\alpha}{}_B]-\frac{1}{2}\left(\bar{\Sigma}^a\right)_{AB}\Lambda^{\alpha A}[\varphi_a,\Lambda^B_\alpha]\right].
\end{aligned}
\tag{2.5}
$$

Here the four-dimensional Euclidean sigma matrices are $\sigma_\mu = (i\tau^1, i\tau^2, i\tau^3, 1)$ and $\bar{\sigma}_\mu = (-i\tau^1, -i\tau^2, -i\tau^3, 1)$, where τ^i $(i = 1, 2, 3)$ are the Pauli matrices. The six-dimensional sigma matrices are given by

$$
\Sigma^a = \left(\eta^3, -i\bar{\eta}^3, \eta^2, -i\bar{\eta}^2, \eta^1, i\bar{\eta}^1\right), \quad \bar{\Sigma}^a = (-\eta^3, -i\bar{\eta}^3, -\eta^2, -i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1), \tag{2.6}
$$

where $a = 5, \cdots, 10$. $\eta^i_{\mu\nu}$ and $\bar{\eta}^i_{\mu\nu}$ are 't Hooft symbols, which are defined by $\sigma_{\mu\nu} = \frac{i}{2}\eta^i_{\mu\nu}\tau^i$ and $\bar{\sigma}_{\mu\nu} = \frac{i}{2}\bar{\eta}^i_{\mu\nu}\tau^i$. After integrating out the auxiliary fields, we get the quartic interaction terms including the gauge fields and scalars, which is given by

$$
\begin{aligned}
\mathcal{L}^{(0)}_{\mathcal{N}=4} \;=\; & \frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\Bigg[-\frac{1}{4}F^{\mu\nu}\left(F_{\mu\nu}+\tilde{F}_{\mu\nu}\right)-i\Lambda^{\alpha A}(\sigma^\mu)_{\alpha\dot\beta}D_\mu\bar{\Lambda}^{\dot\beta}{}_A-\frac{1}{2}\left(D_\mu\varphi_a\right)^2 \\
& +\frac{1}{2}\left(\Sigma^a\right)^{AB}\bar{\Lambda}_{\dot\alpha A}[\varphi_a,\bar{\Lambda}^{\dot\alpha}{}_B]+\frac{1}{2}\left(\bar{\Sigma}^a\right)_{AB}\Lambda^{\alpha A}[\varphi_a,\Lambda^B_\alpha]+\frac{1}{4}[\varphi_a,\varphi_b]^2\Bigg].
\end{aligned}
\tag{2.7}
$$

We call $\mathcal{L}^{(0)}_{\mathcal{N}=4}$ undeformed Lagrangian.

We then introduce a R-R closed string vertex operator

$$V_{\mathcal{F}}^{(-1/2,-1/2)}(z,\bar{z}) = (2\pi\alpha')\mathcal{F}^{\alpha\beta AB}\left[S_\alpha(z)S_A(z)e^{-\frac{1}{2}\phi(z)}S_\beta(\bar{z})S_B(\bar{z})e^{-\frac{1}{2}\phi(\bar{z})}\right] \tag{2.8}$$

with constant $\mathcal{F}^{\alpha\beta AB}$ and insert this vertex operator in a disk amplitude. Here we have used the doubling trick for the spin fields in (2.8) and have replaced the right-moving part in the R-R vertex operator by $S_\beta(\bar{z})S_B(\bar{z})e^{-\frac{1}{2}\phi(\bar{z})}$. The disk amplitude is now given by

$$\langle\!\langle V_{X_1}^{(q_1)}\cdots V_{\mathcal{F}}^{(-\frac{1}{2},-\frac{1}{2})}\cdots\rangle\!\rangle = C_{D_2}\int\frac{\prod_{i=1}^{n}dy_i\prod_{j=1}^{n_{\mathcal{F}}}dz_jd\bar{z}_j}{dV_{CKG}}\langle V_{X_1}^{(q_1)}(y_1)\cdots V_{\mathcal{F}}^{(-\frac{1}{2},-\frac{1}{2})}(z_1,\bar{z}_1)\cdots\rangle, \tag{2.9}$$

where $V_{X_k}^{(q_k)}$ is the open string vertex operator corresponding to a field X_k with picture number q_k, $C_{D_2} = \frac{1}{2\pi^2(\alpha')^2}\frac{1}{kg_{\rm YM}^2}$ is a normalization factor and dV_{CKG} is an $SL(2,\mathbf{R})$-invariant volume factor to fix positions of three coordinates in y_i, z_j and $\bar{z}_j$. The sum of picture numbers in a disk amplitude must be -2.

The constant R-R field strength $\mathcal{F}^{\alpha\beta AB}$ is decomposed into the types $\mathcal{F}^{(\alpha\beta)(AB)}$, $\mathcal{F}^{[\alpha\beta](AB)}$, $\mathcal{F}^{(\alpha\beta)[AB]}$ and $\mathcal{F}^{[\alpha\beta][AB]}$, which are called (S,S), (A,S), (S,A) and (A,A)-type, respectively. It is shown in [19] that the (S,S)-type background corresponds to the R-R 5-form and the (A,S) and (S,A)-types to 3-forms and its dual 7-forms, the (A,A)-type to the 1-form and its dual 9-form. In order to discuss the zero-slope limit, we need to specify the scaling condition for $\mathcal{F}$. In the previous paper [19], we have studied the (S,S)-type deformation with the scaling condition $(2\pi\alpha)^{3/2}\mathcal{F}$ fixed, which would correspond to the deformation of underlying $\mathcal{N}=4$ extended superspace.

In this paper we will consider the (S,A) and (A,S)-type deformations with different scaling condition $(2\pi\alpha')^{1/2}\mathcal{F}$ fixed. These types of deformations cannot be realized by introducing non(anti)commutativity of superspace and give new types of deformed theories. The scaling condition $\mathcal{F}\sim(\alpha')^{-1/2}$ is particularly interesting because it provides the (S,A)-type deformation of D(-1)-instanton effective action similar to the Ω-background in $\mathcal{N}=2$ super Yang-Mills theory [3, 15]. We will consider the effects of the R-R 3-form field strength of (S,A) and (A,S)-types to the low-energy effective Lagrangian in the $\mathcal{N}=4$ case.

2.1 (S,A)-type deformation

2.1.1 Lagrangian

Firstly we discuss the (S,A)-type deformation of $\mathcal{N}=4$ super Yang-Mills theory. For the (S,A)-type background $\mathcal{F}^{(\alpha\beta)[AB]}$, we find that the disk amplitudes which are nonzero in the zero-slope limit are given by $\langle\langle V_A V_\varphi V_\mathcal{F}\rangle\rangle$, $\langle\langle V_{H_{AA}} V_\varphi V_\mathcal{F}\rangle\rangle$ and $\langle\langle V_\Lambda V_\Lambda V_\mathcal{F}\rangle\rangle$. The explicit computations of these amplitudes are essentially the same as in [19]. We do not repeat detailed calculations here. The first two amplitudes become

$$\begin{aligned}&\langle\langle V_A^{(0)}(p_1)V_\varphi^{(-1)}(p_2)V_\mathcal{F}^{(-1/2,-1/2)}\rangle\rangle\\&= -(-i)\frac{4\pi}{kg_{\rm YM}^2}\mathrm{Tr}\left[(\sigma^{\mu\nu})_{\alpha\beta}ip_{1\mu}A_\nu(p_1)(\bar{\Sigma}^a)_{AB}\varphi_a(p_2)\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]},\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\langle\langle V_{H_{AA}}^{(0)}(p_1)V_\varphi^{(-1)}(p_2)V_\mathcal{F}^{(-1/2,-1/2)}\rangle\rangle\\&= -(-i)\frac{1}{2i}\frac{1}{2}\frac{4\pi}{kg_{\rm YM}^2}\mathrm{Tr}\left[(\sigma^{\mu\nu})_{\alpha\beta}H_{\mu\nu}(p_1)(\bar{\Sigma}^a)_{AB}\varphi_a(p_2)\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}.\end{aligned}\tag{2.11}$$

The interaction terms corresponding to these amplitudes are given by

$$-\frac{2\pi i}{kg_{\rm YM}^2}\mathrm{Tr}\left[(\sigma^{\mu\nu})_{\alpha\beta}\left(\partial_{[\mu}A_{\nu]}-\frac{i}{2}H_{\mu\nu}\right)(\bar{\Sigma}^a)_{AB}\varphi_a\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}.\tag{2.12}$$

The third amplitude is

$$\begin{aligned}&\langle\langle V_\Lambda^{(-1/2)}(p_1)V_\Lambda^{(-1/2)}(p_2)V_\mathcal{F}^{(-1/2,-1/2)}\rangle\rangle\\&= i\frac{4\pi i}{kg_{\rm YM}^2}\mathrm{Tr}\left[\varepsilon_{ABCD}\Lambda_\alpha{}^A(p_1)\Lambda_\beta{}^B(p_2)\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[CD]}.\end{aligned}\tag{2.13}$$

Introducing symmetric factor in (2.13) and adding the terms (2.12), we obtain the interaction term including auxiliary fields. Integrating out the auxiliary fields, we find that the deformed Lagrangian is $\mathcal{L}^{(0)}_{\mathcal{N}=4}+\mathcal{L}^{(1)}_{\rm (S,A)}+\mathcal{L}^{(2)}_{\rm (S,A)}+\cdots$, where

$$\mathcal{L}^{(1)}_{\rm (S,A)} = \frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[iF_{\mu\nu}\varphi_a\right]C^{\mu\nu a}-\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\varepsilon_{ABCD}\Lambda_\alpha{}^A\Lambda_\beta{}^B\right]C^{(\alpha\beta)[CD]},\tag{2.14}$$

$$\mathcal{L}^{(2)}_{\rm (S,A)} = \frac{1}{2}\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\varphi_a\varphi_b\right]C_{\mu\nu}{}^aC^{\mu\nu b}.\tag{2.15}$$

Here we have defined the deformation parameter by

$$\begin{aligned}C^{\mu\nu a} &\equiv -2\pi(2\pi\alpha')^{\frac{1}{2}}(\sigma^{\mu\nu})_{\alpha\beta}\left(\bar{\Sigma}^a\right)_{AB}\mathcal{F}^{(\alpha\beta)[AB]},\\C^{(\alpha\beta)[AB]} &\equiv -2\pi(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}.\end{aligned}\tag{2.16}$$

The $O(C^2)$ term $\mathcal{L}^{(2)}_{(\mathrm{S,A})}$ arises from the integration over the auxiliary field. It is possible to construct higher order $O(C^n)$ terms from the disk amplitudes. It is not clear that these amplitudes are reducible or not. For example, at order C^2, there is an amplitude $\langle\langle V_{H_{\varphi\varphi}} V_{\mathcal{F}} V_{\mathcal{F}} \rangle\rangle$, which might change the coefficients of the $\varphi^2 C^2$ term in $\mathcal{L}^{(2)}_{(\mathrm{S,A})}$. However, as we will see in section 4, the reduction from $\mathcal{N}=4$ to $\mathcal{N}=2$ theory shows that $\mathcal{L}^{(2)}_{(\mathrm{S,A})}$ gives the $O(C^2)$ term of the $\mathcal{N}=2$ theory, where the $O(C^2)$ term is exact. Moreover, as we see in the next subsection, the $O(C^2)$ deformed Lagrangian is invariant under $O(C)$ deformed supersymmetry for some C. This is rather different from non(anti)commutative $\mathcal{N}=2$ supersymmetric gauge theory, where deformed supersymmetry transformation contains higher order contributions of the deformation parameter [27]. In order to cancel this deformation transformation, it is necessary to introduce infinite number of interaction terms. But for the (S,A)-deformed Lagrangian we do not need to introduce such a higher order counter term. These properties suggest that the deformed Lagrangian $\mathcal{L}^{(0)}_{\mathcal{N}=4} + \mathcal{L}^{(1)}_{(\mathrm{S,A})} + \mathcal{L}^{(2)}_{(\mathrm{S,A})}$ is an exact Lagrangian, which would be difficult to prove in the NSR formalism.

2.1.2 Deformed Supersymmetry

We examine supersymmetry of the deformed Lagrangian. The Lagrangian $\mathcal{L}^{(0)}_{\mathcal{N}=4}$ of $\mathcal{N}=4$ super Yang-Mills theory is invariant under on-shell $\mathcal{N}=4$ supersymmetry, which is

$$
\begin{aligned}
\delta_0 A_\mu &= i(\xi^A \sigma_\mu \bar{\Lambda}_A + \bar{\xi}_A \bar{\sigma}_\mu \Lambda^A), \\
\delta_0 \Lambda^A &= \sigma^{\mu\nu} \xi^A F_{\mu\nu} + (\Sigma_a)^{AB} \sigma^\mu \bar{\xi}_B D_\mu \varphi_a - i(\Sigma_{ab})^A{}_B \xi^B [\varphi_a, \varphi_b], \\
\delta_0 \bar{\Lambda}_A &= \bar{\sigma}^{\mu\nu} \bar{\xi}_A F_{\mu\nu} + (\bar{\Sigma}_a)_{AB} \bar{\sigma}^\mu \xi^B D_\mu \varphi_a - i(\bar{\Sigma}_{ab})_A{}^B \bar{\xi}_B [\varphi_a, \varphi_b], \\
\delta_0 \varphi_a &= i(\xi^A (\bar{\Sigma}_a)_{AB} \Lambda^B + \bar{\xi}_A (\Sigma_a)^{AB} \bar{\Lambda}_B).
\end{aligned}
\tag{2.17}
$$

The deformed Lagrangian $\mathcal{L}^{(0)}_{\mathcal{N}=4} + \mathcal{L}^{(1)}_{(\mathrm{S,A})} + \mathcal{L}^{(2)}_{(\mathrm{S,A})} + \cdots$ is not invariant under this supersymmetry. We explore deformation of supersymmetry under which the deformed Lagrangian is invariant. The deformed supersymmetry transformation δ can be expanded in the form $\delta = \delta_0 + \delta_1 + \cdots$, where δ_n is the variation including of the n-th order power of C. The deformed supersymmetry δ_n is determined recursively by solving the conditions [28, 29]

$$
\delta_1 \mathcal{L}^{(0)}_{\mathcal{N}=4} + \delta_0 \mathcal{L}^{(1)}_{(\mathrm{S,A})} = 0, \quad \delta_2 \mathcal{L}^{(0)}_{\mathcal{N}=4} + \delta_1 \mathcal{L}^{(1)}_{(\mathrm{S,A})} + \delta_0 \mathcal{L}^{(2)}_{(\mathrm{S,A})} = 0,
\tag{2.18}
$$

and so on. However, we find that there is no solution of (2.18) for generic C. In the first equation of (2.18), a part of the variation $\delta_0\mathcal{L}^{(1)}_{(\mathrm{S,A})}$ is canceled by deforming the supersymmetry transformation of Λ^A as

$$\delta_1\Lambda^A = -i\varphi_a C_{\mu\nu a}\sigma^{\mu\nu}\xi^A. \tag{2.19}$$

Then, at the first order in C, we have

$$\begin{aligned}
\delta_1\mathcal{L}^{(0)}_{\mathcal{N}=4} + \delta_0\mathcal{L}^{(1)}_{(\mathrm{S,A})} = {}& \frac{1}{kg^2_{\mathrm{YM}}}\mathrm{Tr}\Big[-C^{(\alpha\beta)a}(\bar{\Sigma}_a)_{AB}\xi^A_\beta F_{\mu\nu}(\sigma^{\mu\nu}\Lambda^B)_\alpha \\
&\quad - iC^{(\alpha\beta)a}(\bar{\Sigma}_{bc})_A{}^B(\bar{\Sigma}_a)_{BC}\xi^C_\alpha[\varphi_b,\varphi_c]\Lambda^A_\beta\Big] \\
&+\frac{1}{kg^2_{\mathrm{YM}}}\mathrm{Tr}\Big[-F_{\mu\nu}C^{\mu\nu a}\bar{\xi}_A(\Sigma_a)^{AB}\bar{\Lambda}_B \\
&\quad + C^{(\alpha\beta)a}\varphi_b(\bar{\Sigma}_b)_{BA}(\Sigma_a)^{AC}(\sigma^\mu\bar{\xi}_C)_\alpha D_\mu\Lambda^B_\beta\Big].
\end{aligned} \tag{2.20}$$

In order that the supersymmetry variation (2.20) vanishes, we have to require

$$\varepsilon_{ABCD}C^{(\alpha\beta)[BC]}\xi^D_\beta = 0, \quad C^{(\alpha\beta)[AB]}\bar{\xi}_{\dot{\alpha}B} = 0, \tag{2.21}$$

which have only a trivial solution $\xi = \bar{\xi} = 0$ for generic C. The variation of the second order in C also vanishes by the same condition without introducing δ_2. For special C such that (2.21) have nontrivial solution, the theory is invariant under the deformed supersymmetry $\delta = \delta_0 + \delta_1$ at the second order in C. Although we do not fully classify the unbroken supersymmetries in this paper, we illustrate the number of deformed supersymmetry in the case where only $C^{(\alpha\beta)[12]}$ and $C^{(\alpha\beta)[34]}$ are nonzero. From (2.21) the number of unbroken supersymmetry depends on the rank of $C^{(\alpha\beta)[12]}$ and $C^{(\alpha\beta)[34]}$. We summarize the number of unbroken supersymmetries in table 1, where $\mathcal{N} = (p/2, q/2)$ denotes supersymmetry with p chiral and q anti-chiral supercharges.

2.1.3 Deformed scalar potential

In the case of non(anti)commutative $\mathcal{N} = 4$ super Yang-Mills theory, fuzzy sphere configuration with the constant $U(1)$ gauge field background is found [22, 19]. In the deformed Lagrangian (2.14)-(2.15), the scalar potential receives also corrections from the R-R background. We investigate how classical vacua configuration is deformed.

		rank of $C^{(\alpha\beta)[12]}$		
		0	1	2
rank of $C^{(\alpha\beta)[34]}$	0	$\mathcal{N}=(2,2)$	$\mathcal{N}=(3/2,1)$	$\mathcal{N}=(1,1)$
	1	$\mathcal{N}=(3/2,1)$	$\mathcal{N}=(1,0)$	$\mathcal{N}=(1/2,0)$
	2	$\mathcal{N}=(1,1)$	$\mathcal{N}=(1/2,0)$	$\mathcal{N}=(0,0)$

Table 1: The number of unbroken supersymmetry in $\mathcal{N}=4$ SYM with (S,A)-type deformation in the case where only $C^{(\alpha\beta)[12]}$ and $C^{(\alpha\beta)[34]}$ are nonzero.

The scalar potential reads

$$V(\varphi) = -\frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\frac{1}{4}[\varphi_a,\varphi_b]^2 + \frac{1}{2}(C^{\mu\nu a}\varphi_a)^2\right]. \tag{2.22}$$

The stationary condition becomes

$$-\frac{\partial V(\varphi)}{\partial \varphi_a} = \Big[\varphi_b,[\varphi_a,\varphi_b]\Big] + C_{\mu\nu a}C^{\mu\nu b}\varphi_b = 0. \tag{2.23}$$

We explore the solution with the fuzzy sphere ansatz such as

$$[\varphi_a,\varphi_b] = if_{abc}\varphi_c, \tag{2.24}$$

where f_{abc} is a constant antisymmetric tensor. If we plug (2.24) into (2.23), we obtain

$$f_{abc}f_{bcd} = C_{\mu\nu a}C^{\mu\nu}{}_d. \tag{2.25}$$

Hence (2.24) is a solution of (2.23) if (2.25) is satisfied. We regard $C^{\mu\nu}{}_a$ as a 6×6 matrix of which rows and columns are specified by $\mu\nu$ and a respectively. One can find the rank of $C^{\mu\nu}{}_a$ is three due to the self-dual condition. The rank of $C_{\mu\nu a}C^{\mu\nu}{}_d$ is also three. Then we can take the basis such that the upper-left 3×3 submatrix of $C_{\mu\nu a}C^{\mu\nu}{}_d$ is only nonzero. The solution becomes

$$[\varphi_a,\varphi_b] = if_{abc}\varphi_c \qquad \text{for } a,b,c=5,6,7, \tag{2.26}$$

$$[\varphi_a,\varphi_b] = 0 \qquad\qquad \text{otherwise}, \tag{2.27}$$

where f_{abc} is totally antisymmetric tensor. After the appropriate rescaling of φ_a, (2.26) becomes the $SU(2)$ algebra[1]. Therefore (2.26) gives the fuzzy $\mathcal{S}^2$ solution. We note that this fuzzy sphere configuration arises without the constant $U(1)$ gauge field strength background, which is different from non(anti)commutative case [22, 19].

[1]We assume that f_{abc} are real.

2.2 (A,S)-type deformation

2.2.1 Lagrangian

In the (A,S)-type background, nonzero amplitudes with one graviphoton vertex operator are given by $\langle\langle V_{H_{\varphi\varphi}} V_\varphi V_{\mathcal{F}} \rangle\rangle$ and $\langle\langle V_{\bar\Lambda} V_{\bar\Lambda} V_{\mathcal{F}} \rangle\rangle$, which are evaluated as

$$\begin{aligned}&\langle\langle V^{(0)}_{H_{\varphi\varphi}}(p_1) V^{(-1)}_{\varphi}(p_2) V^{(-1/2,-1/2)}_{\mathcal{F}} \rangle\rangle \\ &= -\frac{1}{\sqrt{2}} \frac{\pi i}{k g_{\rm YM}^2} \mathrm{Tr}\left[(\bar\Sigma^a \Sigma^b \bar\Sigma^c)_{AB} H_{ab}(p_1) \varphi_c(p_2)\right] (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}^{[\alpha\beta](AB)} \varepsilon_{\alpha\beta}, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned}&\langle\langle V^{(-1/2)}_{\bar\Lambda}(p_1) V^{(-1/2)}_{\bar\Lambda}(p_2) V^{(-1/2,-1/2)}_{\mathcal{F}} \rangle\rangle \\ &= \frac{4\pi i}{k g_{\rm YM}^2} \mathrm{Tr}\left[\bar\Lambda_{\dot\alpha A}(p_1) \bar\Lambda^{\dot\alpha}{}_B(p_2)\right] (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}^{[\alpha\beta](AB)} \varepsilon_{\alpha\beta}. \end{aligned} \tag{2.29}$$

After including the symmetric factor 1/2! for the second amplitude, we find that new interaction terms induced by the (A,S)-type background are

$$-\frac{1}{\sqrt{2}} \frac{1}{k g_{\rm YM}^2} \mathrm{Tr}\left[(\bar\Sigma^a \Sigma^b \bar\Sigma^c)_{AB} H_{ab} \varphi_c\right] C^{(AB)} + \frac{2}{k g_{\rm YM}^2} \mathrm{Tr}\left[\bar\Lambda_{\dot\alpha A} \bar\Lambda^{\dot\alpha}{}_B\right] C^{(AB)}, \tag{2.30}$$

where

$$C^{(AB)} \equiv -\pi i (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}^{[\alpha\beta](AB)} \varepsilon_{\alpha\beta}. \tag{2.31}$$

After integrating out the auxiliary fields, the deformed Lagrangian is written as $\mathcal{L}^{(0)}_{\mathcal{N}=4} + \mathcal{L}^{(1)}_{\rm (A,S)} + \mathcal{L}^{(2)}_{\rm (A,S)}$, where

$$\mathcal{L}^{(1)}_{\rm (A,S)} = \frac{1}{k g_{\rm YM}^2} \mathrm{Tr}\left[(\bar\Sigma^a \Sigma^b \bar\Sigma^c)_{AB} \varphi_a \varphi_b \varphi_c\right] C^{(AB)} + \frac{2}{k g_{\rm YM}^2} \mathrm{Tr}\left[\bar\Lambda_{\dot\alpha A} \bar\Lambda^{\dot\alpha}_B\right] C^{(AB)}, \tag{2.32}$$

$$\mathcal{L}^{(2)}_{\rm (A,S)} = \frac{1}{4} \frac{1}{k g_{\rm YM}^2} \mathrm{Tr}\left[(\bar\Sigma^a \Sigma^b \bar\Sigma^c)_{AB} (\bar\Sigma^a \Sigma^b \bar\Sigma^d)_{CD} \varphi_c \varphi_d\right] C^{(AB)} C^{(CD)}. \tag{2.33}$$

Here $\mathcal{L}^{(2)}_{\rm (A,S)}$ arises by integration over the auxiliary fields. In contrast to the (S,A)-type deformation, there are no other nonzero open string disk amplitudes at $O(C^2)$ in the case of (A,S)-type background. Therefore the $O(C^2)$ term is exact although there might exist higher order deformed terms.

2.2.2 Deformed supersymmetry

We study supersymmetry of the deformed Lagrangian. As in the case of (S,A)-type deformation, we expand supersymmetry transformation as $\delta = \delta_0 + \delta_1 + \cdots$. Then the variation of the deformed Lagrangian at the first order in C is

$$\delta_1 \mathcal{L}^{(0)}_{\mathcal{N}=4} + \delta_0 \mathcal{L}^{(1)}_{(\mathrm{A,S})} = \frac{1}{kg_{\mathrm{YM}}^2} \mathrm{Tr}\Big[6iC^{(AB)} (\bar{\Sigma}^{ab})_B{}^C \varepsilon_{ACDE} \xi^E \Lambda^D - 4C^{(AB)} \bar{\xi}_A \bar{\sigma}^{\mu\nu} \bar{\Lambda}_B F_{\mu\nu} + iC^{(AB)} (\bar{\Sigma}^{ab})_B{}^C (2\bar{\xi}_C \bar{\Lambda}_A - 6\bar{\xi}_A \bar{\Lambda}_C) [\varphi_a, \varphi_b] \Big], \tag{2.34}$$

where we have deformed the supersymmetry transformation of Λ^A as

$$\delta_1 \Lambda^A = -4iC^{(AB)} (\bar{\Sigma}^a)_{BC} \xi^C \varphi_a. \tag{2.35}$$

The supersymmetry variation (2.34) vanishes if $C^{(AB)}$ satisfies

$$\begin{aligned} &C^{(AB)} (\bar{\Sigma}^{ab})_B{}^C \varepsilon_{ACDE} \xi^E = 0, \\ &C^{(AB)} \bar{\xi}_B = 0, \quad C^{(AB)} (\bar{\Sigma}^{ab})_B{}^C \bar{\xi}_C = 0. \end{aligned} \tag{2.36}$$

The variation of second order in $C^{(AB)}$ also vanishes by the same condition without introducing δ_2. If the rank of $C^{(AB)}$ is one, we have one nonzero ξ^A and no nonzero $\bar{\xi}_A$ as the solution of (2.36). Then the supersymmetry is broken to $\mathcal{N} = (1/2, 0)$. If the rank of $C^{(AB)}$ is more than one, all supersymmetries are broken.

2.2.3 Deformed scalar potential

In the case of (A,S)-type deformation, the potential for the adjoint scalar field is

$$\begin{aligned} -V(\varphi) &= \frac{1}{4} [\varphi_a, \varphi_b]^2 + (\bar{\Sigma}^a \Sigma^b \bar{\Sigma}^c)_{AB} \varphi_a \varphi_b \varphi_c C^{(AB)} \\ &\quad + \frac{1}{4} (\bar{\Sigma}^a \Sigma^b \bar{\Sigma}^c)_{AB} (\bar{\Sigma}^a \Sigma^b \bar{\Sigma}^d)_{CD} \varphi_c \varphi_d C^{(AB)} C^{(CD)}. \end{aligned} \tag{2.37}$$

The stationary condition is

$$\begin{aligned} -\frac{\partial V}{\partial \varphi_a} &= [\varphi_b, [\varphi_a, \varphi_b]] + \frac{3}{2} (\bar{\Sigma}^a \Sigma^b \bar{\Sigma}^c)_{AB} C^{(AB)} [\varphi_b, \varphi_c] \\ &\quad + \frac{1}{2} (\bar{\Sigma}^a \Sigma^b \bar{\Sigma}^c)_{AB} (\bar{\Sigma}^b \Sigma^c \bar{\Sigma}^d)_{CD} C^{(AB)} C^{(CD)} \varphi_d = 0. \end{aligned} \tag{2.38}$$

This equation has a fuzzy sphere solution. Let us assume that φ_a satisfies the commutation relation

$$[\varphi_a, \varphi_b] = i\alpha f_{abc}\varphi_c, \tag{2.39}$$

where $f^{abc} = (\bar{\Sigma}^a\Sigma^b\bar{\Sigma}^c)_{AB}C^{(AB)}$. The constant α is fixed by the equation

$$\left(\alpha^2 - \frac{3}{2}i\alpha - \frac{1}{2}\right) f_{abc}f_{bcd}\varphi_d = 0, \tag{2.40}$$

which are obtained by the substitution of (2.39) into the stationary condition (2.38). The equation (2.40) admits nonzero solutions. Therefore we can formally obtain the nontrivial fuzzy sphere solutions. However, f_{abc} is subjected by the (imaginary) self-dual condition

$$f_{abc} = \frac{i}{3!}\varepsilon_{abcdef}f_{def}. \tag{2.41}$$

For instance, if $f_{5,6,7}$ is real, $f_{8,9,10}$ is imaginary. We should consider the fuzzy sphere configuration in the complexified space of the scalar fields.

3 Non-abelian Chern-Simons term

In this section, we will check that the new bosonic interaction terms arising from the (S,A) and (A,S)-type backgrounds are consistent with the non-abelian Chern-Simons term in the D-brane effective action [21]. The Chern-Simons term is written as

$$S_{CS} \quad = \quad \frac{\mu_3}{k}\mathrm{STr}\int_{\mathcal{M}_4}\sum_n P[e^{i\lambda \mathrm{i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(n)}]e^{\lambda F}. \tag{3.1}$$

Here $\lambda = 2\pi\alpha'$, $\mathcal{A}^{(n)}$ is an n-form R-R potential, $\mu_3 = \frac{1}{\lambda^2 g_{\mathrm{YM}}^2}$ is the R-R charge of a D3-brane. The integral is performed over the four-dimensional D3-brane worldvolume $\mathcal{M}_4$. $F = \frac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu$ is a $U(N)$ gauge field strength which lives in the D3-brane worldvolume and φ_a is $U(N)$ adjoint scalar fields. The symbol P denotes the pull-back of ten-dimensional fields and i_φ is the interior product by φ^a. STr is a symmetric trace of $U(N)$ gauge group. In the following, we will take a static gauge in which the four-dimensional part in ten-dimensional space-time is identified with the worldvolume direction.

3.1 (S,A)-type deformation

For the (S,A)-type background, there exists the R-R 3-form and its dual 7-form field strength with the index structure

$$\begin{aligned} \mathcal{F}_{\mu\nu a} &= \partial_{[\mu}\mathcal{A}_{\nu]a} + \partial_a \mathcal{A}_{\mu\nu}, \\ \mathcal{F}_{\mu\nu abcde} &= \partial_{[\mu}\mathcal{A}_{\nu]abcde} + \partial_{(a}\mathcal{A}_{bcde)\mu\nu}, \end{aligned} \tag{3.2}$$

where $\mu,\nu = 1,\ldots,4$ are worldvolume directions and $a,b,\ldots,e = 5,\ldots,10$ are six-dimensional directions transverse to the D-brane worldvolume.

First, we calculate contributions from the 3-form field strength with the 2-form potentials $\mathcal{A}_{\mu\nu}, \mathcal{A}_{\mu a}$. The Chern-Simons term is

$$\frac{\mu_3}{k}\mathrm{STr}\int_{\mathcal{M}_4} P[e^{i\lambda \mathrm{i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(2)}]e^{\lambda F}\bigg|_{(\mathrm{S,A})} = \frac{\mu_3}{k}\mathrm{STr}\frac{\lambda}{4}\int_{\mathcal{M}_4} P[\mathcal{A}^{(2)}]_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}d^4x. \tag{3.3}$$

Here $|_{(\mathrm{S,A})}$ means the restriction of the R-R indices to (S,A)-type deformation (3.2). The pull-back is given by

$$P[\mathcal{A}^{(2)}]_{\mu\nu} = \mathcal{A}_{MN}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = \mathcal{A}_{\mu\nu} + 2\lambda\mathcal{A}_{\mu a}D_\nu\varphi_a. \tag{3.4}$$

Here X^M $(M = (\mu, a) = 1,\ldots,10)$ are ten-dimensional space-time coordinates where X^a are identified with adjoint scalar fields in $\mathcal{N} = 4$ vector multiplet through $X^a = \lambda\varphi_a$. Note that the pull-back is covariantized with respect to $U(N)$ gauge group. The potential has to be expanded by the fluctuation φ_a such that

$$\begin{aligned} \mathcal{A}_{\mu\nu} &= \mathcal{A}^{(0)}_{\mu\nu} + \lambda\varphi_c\partial_c\mathcal{A}^{(0)}_{\mu\nu}, \\ \mathcal{A}_{\mu a} &= \mathcal{A}^{(0)}_{\mu a}. \end{aligned} \tag{3.5}$$

Here $\mathcal{A}^{(0)}_{\mu\nu}, \mathcal{A}^{(0)}_{\mu a}$ are 2-form potentials evaluated at $\varphi_a = 0$. In the following we omit the superscript (0). After using Bianchi identity $\varepsilon^{\mu\nu\rho\sigma}D_\nu F_{\rho\sigma} = 0$ and partial integrations, we find

$$\frac{\mu_3}{k}\mathrm{STr}\int_{\mathcal{M}_4} P[e^{i\lambda \mathrm{i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(2)}]e^{\lambda F}\bigg|_{(\mathrm{S,A})} = \frac{1}{2kg^2_{\mathrm{YM}}}\int_{\mathcal{M}_4} d^4x\ \mathrm{Tr}\left[\varphi_a F_{\mu\nu}\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{\mu\nu a}. \tag{3.6}$$

By identifying $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{\mu\nu a} = 2iC^{\mu\nu a}$, this Chern-Simons term precisely agrees with the $O(C)$ part of the (S,A)-deformation term (2.14).

Next, we calculate contributions from the 7-form part, which take the form

$$\frac{\mu_3}{k}\mathrm{STr}\int_{\mathcal{M}_4} P[e^{i\lambda \mathrm{i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(6)}]e^{\lambda F}$$
$$= \frac{\mu_3}{k}\mathrm{STr}\int_{\mathcal{M}_4}\left[i\lambda^{\frac{3}{2}}P[\mathrm{i}_\varphi^2\mathcal{A}^{(6)}] - \frac{1}{2}\lambda^{\frac{7}{2}}P[(\mathrm{i}_\varphi^2)^2\mathcal{A}^{(6)}]\wedge F - \frac{i}{2\cdot 3!}\lambda^{\frac{11}{2}}P[(\mathrm{i}_\varphi^2)^3\mathcal{A}^{(6)}]\wedge F\wedge F\right], \tag{3.7}$$

where $\mathcal{A}^{(6)}$ takes the form either $\mathcal{A}_{\mu abcde}$ or $\mathcal{A}_{\mu\nu abcd}$. After evaluating STr, pull-back, and expansion in fluctuation, we find that (3.7) becomes

$$\frac{\lambda^{\frac{3}{2}}}{kg_{\mathrm{YM}}^2}\int_{\mathcal{M}_4} d^4x\ \varepsilon^{\mu\nu\rho\sigma}\mathrm{STr}\left[\frac{i}{4}\mathcal{A}_{abcd\mu\nu}\varphi_b\varphi_a D_\rho\varphi_c D_\sigma\varphi_d - \frac{1}{8}\mathcal{A}_{abcd\mu\nu}\varphi_d\varphi_c\varphi_b\varphi_a F_{\rho\sigma}\right]$$
$$+\frac{\lambda^{\frac{5}{2}}}{kg_{\mathrm{YM}}^2}\int_{\mathcal{M}_4} d^4x\ \varepsilon^{\mu\nu\rho\sigma}\mathrm{STr}\left[\frac{i}{6}\mathcal{A}_{abcde\mu}\varphi_b\varphi_a D_\nu\varphi_c D_\rho\varphi_d D_\sigma\varphi_e + \frac{i}{4}\partial_e\mathcal{A}_{abcd\mu\nu}\varphi_b\varphi_a\varphi_e D_\rho\varphi_c D_\sigma\varphi_d\right.$$
$$\left.-\frac{1}{8}\partial_e\mathcal{A}_{abcd\mu\nu}\varphi_d\varphi_c\varphi_b\varphi_a\varphi_e F_{\rho\sigma} - \frac{1}{4}\mathcal{A}_{abcde\mu}\varphi_d\varphi_c\varphi_b\varphi_a D_\nu\varphi_e F_{\rho\sigma}\right]. \tag{3.8}$$

The $\lambda^{\frac{3}{2}}$ term vanishes by the partial integration. The $\lambda^{\frac{5}{2}}$ term will not contribute to the deformation term in field theory limit in our scaling $\lambda^{\frac{1}{2}}\mathcal{F} = $ fixed. Thus we see that our open string calculation is consistent with effective action of D-brane in the presence of R-R background for the (S,A)-type deformation.

3.2 (A,S)-type deformation

For the (A,S)-type deformation, the R-R 3-form and its dual 7-form field strength with index structure are given by

$$\mathcal{F}_{abc} = \partial_{(a}\mathcal{A}_{bc)},$$
$$\mathcal{F}_{\mu\nu\rho\sigma abc} = \partial_{[\mu}\mathcal{A}_{\nu\rho\sigma]abc} + \partial_{(a}\mathcal{A}_{bc)\mu\nu\rho\sigma}. \tag{3.9}$$

The Chern-Simons term corresponding to the R-R 2-form potential is

$$\frac{\mu_3}{k}\mathrm{STr}\int_{\mathcal{M}_4} P[e^{i\lambda \mathrm{i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(2)}]e^{\lambda F}\bigg|_{\mathrm{(A,S)}} = \frac{\lambda^{\frac{5}{2}}}{4kg_{\mathrm{YM}}^2}\mathrm{STr}\int_{\mathcal{M}_4} d^4x\ \partial_c\mathcal{A}_{ab}\varphi_c D_\mu\varphi_a D_\nu\varphi_b F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$$
$$+\frac{i\lambda^{\frac{5}{2}}}{8kg_{\mathrm{YM}}^2}\mathrm{STr}\int_{\mathcal{M}_4} d^4x\ \partial_c\mathcal{A}_{ab}\varphi_b\varphi_a\varphi_c F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}. \tag{3.10}$$

After evaluating STr and performing partial integrations, we find that this becomes

$$\frac{1}{12kg_{\rm YM}^2}\int_{\mathcal{M}_4} d^4x \ {\rm Tr}\left[\varphi_a D_\mu\varphi_b D_\nu\varphi_c F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right](2\pi\alpha')^{\frac{5}{2}}\mathcal{F}_{abc}$$
$$-\frac{i}{24kg_{\rm YM}^2}\int_{\mathcal{M}_4} d^4x \ {\rm Tr}\left[\varphi_a\varphi_b\varphi_c F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right](2\pi\alpha')^{\frac{5}{2}}\mathcal{F}_{abc}. \qquad (3.11)$$

Those terms vanish in the zero-slope limit $\alpha' \to 0$ with fixed $\lambda^{\frac{1}{2}}\mathcal{F}$. On the other hand, the 7-form part is calculated by the same way as

$$\frac{\mu_3}{k}{\rm STr}\int_{\mathcal{M}_4} P[e^{i\lambda {\rm i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(6)}]e^{\lambda F}\Big|_{\rm (A,S)} = -\frac{i}{3\cdot 4!}\frac{1}{kg_{\rm YM}^2}\int_{\mathcal{M}_4} d^4x \ {\rm Tr}\left[\varphi_a\varphi_b\varphi_c\right](2\pi\alpha')^{\frac{1}{2}}\tilde{\mathcal{F}}_{abc}. \qquad (3.12)$$

Here we have defined

$$\tilde{\mathcal{F}}_{abc} \equiv \mathcal{F}_{abc\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}. \qquad (3.13)$$

This term precisely agrees with the (A,S)-type deformation term (2.32) at linear order in deformation parameter with the identification $-\frac{i}{3\cdot 4!}(2\pi\alpha')^{\frac{1}{2}}\tilde{\mathcal{F}}^{abc} = (\bar{\Sigma}^a\Sigma^b\bar{\Sigma}^c)_{AB}C^{(AB)}$. Therefore the (A,S)-type deformation is related to the dual 7-form R-R field strength.

4 Deformed $\mathcal{N}=2$ super Yang-Mills theories

So far we have studied the deformation of $\mathcal{N}=4$ super Yang-Mills theory in the R-R 3-form background. In this section we study deformed $\mathcal{N}=2$ $U(N)$ super Yang-Mills theory in the (S,A) and (A,S)-type backgrounds. To realize $\mathcal{N}=2$ $U(N)$ supersymmetric gauge theory, we use N fractional D3-branes located at the singularity of the orbifold $\mathbf{C}^2/\mathbf{Z}_2$[30]. Since the orbifold projection restricts R-symmetry group $SU(4)$ to $SU(2)$, the internal spin fields S_A become the doublet S_i ($i=1,2$) of $SU(2)$. The massless fields on the fractional D3-branes are gauge fields A_μ, Weyl fermions Λ^i_α and a complex scalar φ, whose vertex operators are obtained by the orbifold projection and are defined in [18]. The undeformed Lagrangian is given by

$$\mathcal{L}^{(0)}_{\mathcal{N}=2} = \frac{1}{kg_{\rm YM}^2}{\rm Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - D_\mu\varphi D^\mu\bar{\varphi} - \frac{1}{2}[\varphi,\bar{\varphi}]^2$$
$$-i\Lambda^{i\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}D_\mu\bar{\Lambda}_i{}^{\dot{\beta}} - \frac{i}{\sqrt{2}}\Lambda^i[\bar{\varphi},\Lambda_i] + \frac{i}{\sqrt{2}}\bar{\Lambda}_i[\varphi,\bar{\Lambda}^i]\Big]. \qquad (4.1)$$

We introduce the R-R vertex operator of the form

$$V_{\mathcal{F}}^{(-1/2,-1/2)}(z,\bar{z})=(2\pi\alpha')\mathcal{F}^{\alpha\beta ij}\left[S_\alpha(z)S^{(-)}(z)S_i(z)e^{-\frac{1}{2}\phi(z)}S_\beta(\bar{z})S^{(-)}(\bar{z})S_j(\bar{z})e^{-\frac{1}{2}\phi(\bar{z})}\right].\quad(4.2)$$

The R-R field strength can be decomposed into $\mathcal{F}^{(\alpha\beta)(ij)}$, $\mathcal{F}^{(\alpha\beta)[ij]}$, $\mathcal{F}^{[\alpha\beta](ij)}$ and $\mathcal{F}^{[\alpha\beta][ij]}$, which corresponds to the R-R 5-form, 3-form (7-form), 3-form (7-form) and 1-form (9-form) field strength respectively. We calculate the deformed Lagrangian in the (S,A) and (A,S)-type deformations with the scaling condition $\mathcal{F}\sim(\alpha')^{-1/2}$ as we did in the $\mathcal{N}=4$ case.

4.1 (S,A)-type deformation

The $\mathcal{N}=2$ (S,A)-type deformation was studied in [15]. The nonzero disk amplitudes which contain single $\mathcal{F}^{(\alpha\beta)[ij]}$, are $\langle\langle V_A V_{\bar{\varphi}} V_{\mathcal{F}}\rangle\rangle$ and $\langle\langle V_{H_{AA}} V_{\bar{\varphi}} V_{\mathcal{F}}\rangle\rangle$. The first amplitude is evaluated as

$$\begin{aligned}&\langle\langle V_A^{(0)}(p_1)V_{\bar{\varphi}}^{(-1)}(p_2)V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle\\&=\frac{4\sqrt{2}\pi}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[(\sigma^{\mu\nu})_{\alpha\beta}ip_{1\mu}A_\nu(p_1)\bar{\varphi}(p_2)\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[ij]}\varepsilon_{ij}.\end{aligned}\quad(4.3)$$

Combining the result of the second amplitude, we get the interaction term

$$-(-i)\frac{2\sqrt{2}\pi}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[\left(\partial_{[\mu}A_{\nu]}-\frac{i}{2}H_{\mu\nu}\right)\bar{\varphi}(\sigma^{\mu\nu})_{\alpha\beta}\right]\varepsilon_{ij}\mathcal{F}^{(\alpha\beta)[ij]}.\quad(4.4)$$

After integrating out the auxiliary fields, we find

$$\mathcal{L}_{\mathrm{(S,A)}}^{(1)}+\mathcal{L}_{\mathrm{(S,A)}}^{(2)}=\frac{1}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[iF_{\mu\nu}\bar{\varphi}\tilde{C}^{\mu\nu}+\frac{1}{2}(\bar{\varphi}\tilde{C}^{\mu\nu})^2\right],\quad(4.5)$$

where we have defined $\tilde{C}^{\mu\nu}\equiv 2\sqrt{2}\pi i(\sigma^{\mu\nu})_{\alpha\beta}\varepsilon_{ij}\mathcal{F}^{(\alpha\beta)[ij]}$. Since at order C^2 there are no other disk amplitudes which contribute to the Lagrangian, the deformed Lagrangian is exact up to higher order corrections in C.

The deformation term (4.5) can be also obtained by the reduction from $\mathcal{N}=4$ to $\mathcal{N}=2$ by the $\mathbf{Z}_2$ orbifold projection, which is given by

$$\Lambda_\alpha^A=0 \text{ for } A=3,4,\quad \varphi_a=0 \text{ for } a=7,8,9,10,\quad(4.6)$$

and only $C^{(\alpha\beta)[12]}$ and $C^{(\alpha\beta)[34]}$ are nonzero [15]. Under the reduction, the deformation term becomes

$$\mathcal{L}^{(1)}_{(\mathrm{S,A})}+\mathcal{L}^{(2)}_{(\mathrm{S,A})} = \frac{1}{kg^2_{\mathrm{YM}}}\mathrm{Tr}\left[i(\tilde{C}^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)F_{\mu\nu}-\frac{1}{\sqrt{2}}\bar{C}^{\mu\nu}\Lambda^i\sigma_{\mu\nu}\Lambda_i+\frac{1}{2}(\tilde{C}^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)^2\right], \quad (4.7)$$

where $\tilde{C}^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ are defined as

$$\tilde{C}^{\mu\nu} = 2\sqrt{2}iC^{\mu\nu[12]}, \quad \bar{C}^{\mu\nu} = -2\sqrt{2}iC^{\mu\nu[34]}, \quad (4.8)$$

and we have used

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_5 - i\varphi_6), \quad \bar{\varphi} = \frac{1}{\sqrt{2}}(\varphi_5 + i\varphi_6). \quad (4.9)$$

In the case of $\bar{C}^{\mu\nu} = 0$, (4.7) is reduced to (4.5). The deformation parameter $\bar{C}^{\mu\nu}$ is referred as the graviphoton-like vertex operator in [15].

We examine the deformed supersymmetry of the Lagrangian $\mathcal{L}^{(0)}_{\mathcal{N}=2} + \mathcal{L}^{(1)}_{(\mathrm{S,A})} + \mathcal{L}^{(2)}_{(\mathrm{S,A})}$. The deformed supersymmetry transformation is obtained from the $\mathbf{Z}_2$ projection in $\mathcal{N} = 4$ theory, which is given by

$$\begin{aligned}
\delta A_\mu &= i(\xi^i\sigma_\mu\bar{\Lambda}_i + \bar{\xi}_i\bar{\sigma}_\mu\Lambda^i),\\
\delta\Lambda^i &= \sigma^{\mu\nu}\xi^i\big(F_{\mu\nu} - i(\tilde{C}^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)\big) + \sqrt{2}i\sigma^\mu\bar{\xi}^i D_\mu\varphi - i\xi^i[\varphi,\bar{\varphi}],\\
\delta\bar{\Lambda}_i &= \bar{\sigma}^{\mu\nu}\bar{\xi}_i F_{\mu\nu} - \sqrt{2}i\bar{\sigma}^\mu\xi_i D_\mu\bar{\varphi} + i\bar{\xi}_i[\varphi,\bar{\varphi}],\\
\delta\varphi &= \sqrt{2}\xi^i\Lambda_i,\\
\delta\bar{\varphi} &= \sqrt{2}\bar{\xi}^i\bar{\Lambda}_i.
\end{aligned} \quad (4.10)$$

The deformed Lagrangian is invariant under (4.10) if ξ and $\bar{\xi}$ satisfy

$$\begin{aligned}
&\bar{C}^{(\alpha\beta)}\xi^i_\beta = 0,\\
&\bar{\xi}_i = 0 \text{ or } \tilde{C}^{(\alpha\beta)} = 0,
\end{aligned} \quad (4.11)$$

where $\tilde{C}^{(\alpha\beta)} = 2\sqrt{2}iC^{(\alpha\beta)[12]}$, $\bar{C}^{(\alpha\beta)} = -2\sqrt{2}iC^{(\alpha\beta)[34]}$. As in the $\mathcal{N} = 4$ case, we can classify the unbroken supersymmetries, which are summarized in table 2.

		rank of $\tilde{C}^{(\alpha\beta)}$		
		0	1	2
rank of $\bar{C}^{(\alpha\beta)}$	0	$\mathcal{N}=(1,1)$	$\mathcal{N}=(1,0)$	$\mathcal{N}=(1,0)$
	1	$\mathcal{N}=(1/2,1)$	$\mathcal{N}=(1/2,0)$	$\mathcal{N}=(1/2,0)$
	2	$\mathcal{N}=(0,1)$	$\mathcal{N}=(0,0)$	$\mathcal{N}=(0,0)$

Table 2: The number of unbroken supersymmetry in $\mathcal{N}=2$ SYM with (S,A)-type deformation.

4.2 (A,S)-type deformation

Next we consider the (A,S)-type deformation of $\mathcal{N}=2$ super Yang-Mills theory. At the first order in $\mathcal{F}$, the nonzero amplitude is possible only for

$$\begin{aligned}&\langle\langle V_{\bar{\Lambda}}^{(-1/2)}(p_1)V_{\bar{\Lambda}}^{(-1/2)}(p_2)V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle\\ &\qquad=\frac{4\pi i}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[\bar{\Lambda}_{\dot{\alpha}i}(p_1)\bar{\Lambda}^{\dot{\alpha}}{}_j(p_2)\right](2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{[\alpha\beta](ij)}\varepsilon_{\alpha\beta}.\end{aligned} \tag{4.12}$$

The interaction term is given by

$$\mathcal{L}^{(1)}_{(\mathrm{A,S})}=\frac{1}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[\bar{\Lambda}_{\dot{\alpha}i}(x)\bar{\Lambda}^{\dot{\alpha}}{}_j(x)\right]C^{(ij)}. \tag{4.13}$$

Here $C^{(ij)}\equiv-2\pi i(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{[\alpha\beta](ij)}\varepsilon_{\alpha\beta}$.

As in the case of (S,A)-type deformation, We can obtain deformed Lagrangian from the $\mathcal{N}=4$ one by the reduction. The deformation parameter $C^{(AB)}$ takes the block diagonal form:

$$C^{(AB)}=\frac{1}{2}\begin{pmatrix}C^{(ij)} & 0\\ 0 & C^{(\hat{i}\hat{j})}\end{pmatrix},\quad i,j=1,2,\quad \hat{i},\hat{j}=3,4. \tag{4.14}$$

Then the deformation terms become

$$\mathcal{L}^{(1)}_{(\mathrm{A,S})}+\mathcal{L}^{(2)}_{(\mathrm{A,S})}=\frac{1}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\Big[C^{(ij)}\bar{\Lambda}_{\dot{\alpha}i}\bar{\Lambda}^{\dot{\alpha}}_j-C^{(ij)}C_{(ij)}\bar{\varphi}^2-C^{(\hat{i}\hat{j})}C_{(\hat{i}\hat{j})}\varphi^2\Big]. \tag{4.15}$$

We note that only the $O(C^2)$ terms in (4.15) are allowed to exist at this order due to the charge conservation of vertex operators in the disk amplitudes, which are given by $\langle\langle V_{\bar{\varphi}}V_{\bar{\varphi}}V_{\mathcal{F}}V_{\mathcal{F}}\rangle\rangle$, $\langle\langle V_{\varphi}V_{\varphi}V_{\bar{\mathcal{F}}}V_{\bar{\mathcal{F}}}\rangle\rangle$. Here $V_{\bar{\mathcal{F}}}$ is the closed string R-R vertex operator corresponding to $C^{\hat{i}\hat{j}}$.

The deformed Lagrangian is invariant under the supersymmetry transformation

$$\begin{aligned}
\delta A_\mu &= i(\xi^i\sigma_\mu\bar{\Lambda}_i + \bar{\xi}_i\bar{\sigma}_\mu\Lambda^i),\\
\delta\Lambda^i &= \sigma^{\mu\nu}\xi^i F_{\mu\nu} + \sqrt{2}i\sigma^\mu\bar{\xi}^i D_\mu\varphi - i\xi^i[\varphi,\bar{\varphi}] - 4\sqrt{2}\bar{\varphi}C^{(ij)}\xi_j,\\
\delta\bar{\Lambda}_i &= \bar{\sigma}^{\mu\nu}\bar{\xi}_i F_{\mu\nu} - \sqrt{2}i\bar{\sigma}^\mu\xi_i D_\mu\bar{\varphi} + i\bar{\xi}_i[\varphi,\bar{\varphi}],\\
\delta\varphi &= \sqrt{2}\xi^i\Lambda_i,\\
\delta\bar{\varphi} &= \sqrt{2}\bar{\xi}^i\bar{\Lambda}_i,
\end{aligned} \tag{4.16}$$

if $\bar{\xi}$ satisfies

$$C^{(ij)}\bar{\xi}_j = 0. \tag{4.17}$$

Hence the theory has $\mathcal{N} = (1,0)$ supersymmetry in the generic case. But it is enhanced to $\mathcal{N} = (1,1/2)$ supersymmetry if the rank of $C^{(ij)}$ is one.

4.3 Comments on the reduction to $\mathcal{N} = 1$ theory

We are able to discuss further reduction to deformed $\mathcal{N} = 1$ theory from the orbifold $\mathbf{R}^6/\mathbf{Z}_2\times\mathbf{Z}_2$ [31], which can be done by restriction $\varphi_a = 0$ $(a = 5,\dots 10)$, $\Lambda^{2,3,4} = \bar{\Lambda}^{2,3,4} = 0$ in $\mathcal{N} = 4$ theory. The deformation parameter $\mathcal{F}^{\alpha\beta AB}$ remains nonzero for $A = B = 1$. Therefore it is easy to see that (S,A)-type deformation with parameter $\mathcal{F}^{\alpha\beta[AB]}$ does not exist in $\mathcal{N} = 1$ theory.

On the other hand, the (A,S)-type deformation is still allowed in $\mathcal{N} = 1$ theory. In fact the reduction from $\mathcal{N} = 4$ theory leads to the interaction term

$$\mathcal{L}_{(\mathrm{A,S})} = \frac{1}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[\bar{\Lambda}_{\dot{\alpha}}\bar{\Lambda}^{\dot{\alpha}}\right]C, \tag{4.18}$$

where $C = 2C^{(11)[\alpha\beta]}\varepsilon_{\alpha\beta}$. This result is also consistent with direct computation of string amplitudes.

We note that it is possible to deform $\mathcal{N} = 1$ super Yang-Mills theory in the (S,S)-type background with the scaling condition $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F} =$ fixed, where the scaling condition $(2\pi\alpha')^{\frac{3}{2}}\mathcal{F} =$ fixed leads to a non(anti)commutative deformation of superspace [17, 18, 19]. We find, however, that there are no interaction terms in the zero slope limit from calculation of disk amplitudes and Chern-Simons term. We conclude $\mathcal{N} = 1$ super Yang-Mills theory is not deformed in the (S,S)-type background at least up to leading order in deformation parameter.

5 Deformed Lagrangian in $\mathcal{N}=1$ superspace

Although the (S,A) and (A,S) type deformation is not realized as non(anti)commutative superspace deformation, it would be useful to rewrite the deformation Lagrangian in superfields in order to understand its geometrical structure. In this section we explore a geometrical interpretation of the deformed super Yang-Mills theories in terms of $\mathcal{N}=1$ superspace.

5.1 $\mathcal{N}=4$ deformation

The Lagrangian of $\mathcal{N}=4$ super Yang-Mills theory in $\mathcal{N}=1$ superspace is given by

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=4} &= \frac{1}{kg_{\mathrm{YM}}^2}\int d^2\theta d^2\bar{\theta}\ \mathrm{Tr}\sum_{i=1}^{3}\left(\bar{\Phi}_i e^{2V}\Phi_i e^{-2V}\right)+\frac{1}{16kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[\int d^2\theta\, W^{\alpha}W_{\alpha}+\int d^2\bar{\theta}\,\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right]\\ &\quad -\frac{\sqrt{2}}{3}\frac{1}{kg_{\mathrm{YM}}^2}\int d^2\theta\ \mathrm{Tr}\varepsilon^{ijk}\left(\Phi_i\Phi_j\Phi_k\right)+\frac{\sqrt{2}}{3}\frac{1}{kg_{\mathrm{YM}}^2}\int d^2\bar{\theta}\ \mathrm{Tr}\varepsilon^{ijk}\left(\bar{\Phi}_i\bar{\Phi}_j\bar{\Phi}_k\right).\end{aligned} \tag{5.1}$$

Here $\Phi_i, (\bar{\Phi}_i)$ $(i=1,2,3)$ are (anti-)chiral superfields, V a vector superfield, $W_\alpha, \bar{W}_{\dot{\alpha}}$ its super field strengths, which are written in terms of component fields as

$$\begin{aligned}\Phi_i &= \phi_i(y)+\sqrt{2}\theta\psi_i(y)+\theta\theta F_i(y),\\ \bar{\Phi}_i &= \bar{\phi}_i(\bar{y})+\sqrt{2}\bar{\theta}\bar{\psi}_i(\bar{y})+\bar{\theta}\bar{\theta}\bar{F}_i(\bar{y}),\\ 2^{-1}W_\alpha &= -i\lambda_\alpha+\left[\delta_\alpha{}^\beta D-i(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}\right]\theta_\beta+\theta^2(\sigma^\mu)_{\alpha\dot{\alpha}}D_\mu\bar{\lambda}^{\dot{\alpha}},\\ 2^{-1}\bar{W}_{\dot{\alpha}} &= -i\bar{\lambda}_{\dot{\alpha}}+\left[\varepsilon_{\dot{\alpha}\dot{\beta}}D+i\varepsilon_{\dot{\alpha}\dot{\gamma}}(\bar{\sigma}^{\mu\nu})^{\dot{\gamma}}{}_{\dot{\beta}}F_{\mu\nu}\right]\bar{\theta}^{\dot{\beta}}-\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^2(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}D_\mu\lambda_\alpha.\end{aligned} \tag{5.2}$$

We have followed the notation and convention in [32].

Firstly we consider the (S,A)-type deformation. We can show that the interaction

terms (2.14) and (2.15) are regarded as the deformation of D-terms and F-terms:

$$
\begin{aligned}
&\mathcal{L}^{(1)}_{(\mathrm{S,A})}+\mathcal{L}^{(2)}_{(\mathrm{S,A})}\\
&=\frac{1}{2kg_{\mathrm{YM}}^2}\int d^4\theta\ \theta^2\bar\theta^2\mathrm{Tr}\Big[\left(\bar\Phi_1C^{(\alpha\beta)[12]}+\bar\Phi_2C^{(\alpha\beta)[31]}+\bar\Phi_3C^{(\alpha\beta)[14]}\right)D_\alpha W_\beta\Big]\\
&-\frac{4}{kg_{\mathrm{YM}}^2}\int d^2\theta\ \theta^2\mathrm{Tr}\Big[D_\alpha\Phi_1D_\beta\Phi_2C^{(\alpha\beta)[14]}+D_\alpha\Phi_2D_\beta\Phi_3C^{(\alpha\beta)[12]}+D_\alpha\Phi_3D_\beta\Phi_1C^{(\alpha\beta)[13]}\Big]\\
&+\frac{\sqrt2}{kg_{\mathrm{YM}}^2}\int d^2\theta\ \theta^2\mathrm{Tr}\Big[\left(D_\alpha\Phi_1W_\beta+\Phi_1D_\alpha W_\beta\right)C^{(\alpha\beta)[34]}+\left(D_\alpha\Phi_2W_\beta+\Phi_2D_\alpha W_\beta\right)C^{(\alpha\beta)[24]}\\
&\qquad\qquad+\left(D_\alpha\Phi_3W_\beta+\Phi_3D_\alpha W_\beta\right)C^{(\alpha\beta)[23]}\Big]\\
&+\frac{4}{kg_{\mathrm{YM}}^2}\int d^2\theta\ \theta^2\mathrm{Tr}\Big[(\Phi_1C^{\mu\nu[34]}+\Phi_2C^{\mu\nu[42]}+\Phi_3C^{\mu\nu[23]})^2\Big]\\
&+\frac{4}{kg_{\mathrm{YM}}^2}\int d^2\bar\theta\ \bar\theta^2\mathrm{Tr}\Big[(\bar\Phi_1C^{\mu\nu[12]}+\bar\Phi_2C^{\mu\nu[31]}+\bar\Phi_3C^{\mu\nu[14]})^2\Big].
\end{aligned}
\tag{5.3}
$$

Here we have used the relation

$$
\varphi_{(2i-1)+4}=\frac{1}{\sqrt2}\left(\phi_i+\bar\phi_i\right),\quad \varphi_{2i+4}=\frac{i}{\sqrt2}\left(\phi_i-\bar\phi_i\right).\tag{5.4}
$$

It is natural to think that this complicated expression is simplified if one uses $\mathcal{N}=2$ superspace formalism as in [16], which will be discussed elsewhere.

Next, we study the (A,S)-type deformation. In this case we have simple interpretation of the Lagrangian in terms of deformation of gauge coupling constants and complex mass parameters, which are functions on $\mathcal{N}=1$ superspace. To see this, let us consider generic mass deformation of the $\mathcal{N}=4$ Lagrangian

$$
\mathcal{L}^{\mathcal{N}=4}_m\ =\ \mathcal{L}^{\mathcal{N}=4}+\frac{1}{2kg_{\mathrm{YM}}^2}\int d^2\theta\ \mathrm{Tr}\left(m_i\Phi_i^2\right)+\frac{1}{2kg_{\mathrm{YM}}^2}\int d^2\bar\theta\ \mathrm{Tr}\left(\bar m_i\bar\Phi_i^2\right).\tag{5.5}
$$

In terms of component fields, this is written as

$$
\begin{aligned}
\mathcal{L}^{\mathcal{N}=4}_m=\mathcal{L}^{\mathcal{N}=4}+\frac{1}{kg_{\mathrm{YM}}^2}\mathrm{Tr}\Big[&-|m_i|^2|\phi_i|^2+\sqrt2\varepsilon^{ijk}\bar m_i\bar\phi_i\phi_j\phi_k\\
&-\sqrt2\varepsilon^{ijk}m_i\phi_i\bar\phi_j\bar\phi_k-\frac12 m_i\psi_i^2-\frac12\bar m_i\bar\psi_i^2\Big].
\end{aligned}
\tag{5.6}
$$

The deformation terms (2.32) are written as

$$
\delta\mathcal{L}\ \equiv\ -\frac16 M^{abc}\mathrm{Tr}\Big[\varphi_a\varphi_b\varphi_c\Big]-\frac12\mathrm{Tr}\left[m_{AB}\Lambda^{\alpha A}\Lambda_\alpha{}^B+m^{AB}\bar\Lambda_{\dot\alpha A}\bar\Lambda^{\dot\alpha}{}_B\right],\tag{5.7}
$$

where

$$M^{abc} = m_{AB}(\Sigma^a\bar{\Sigma}^b\Sigma^c)^{AB} + m^{AB}(\bar{\Sigma}^a\Sigma^b\bar{\Sigma}^c)_{AB}, \tag{5.8}$$

and m_{AB} and m^{AB} are 4×4 matrices, which are $m_{AB} = 0$, $m^{AB} = -\frac{1}{4}C^{(AB)}$ in the (A,S)-type deformation [2]

When m_{AB} and m^{AB} take a diagonal form

$$\begin{aligned} m_{AB} &= \mathrm{diag}(-m_0, m_1, m_2, m_3), \\ m^{AB} &= \mathrm{diag}(-\bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3) \end{aligned} \tag{5.9}$$

and $m_0 = \bar{m}_0 = 0$, we find

$$\begin{aligned} \delta\mathcal{L} &\equiv -\frac{1}{6}M^{abc}\mathrm{Tr}\left[\varphi_a\varphi_b\varphi_c\right] - \frac{1}{2}\mathrm{Tr}\left[m_{AB}\Lambda^{\alpha A}\Lambda_\alpha{}^B + m^{AB}\bar{\Lambda}_{\dot{\alpha}A}\bar{\Lambda}^{\dot{\alpha}}{}_B\right] \\ &= \mathrm{Tr}\left[\sqrt{2}\varepsilon^{ijk}\bar{m}_i\bar{\phi}_i\phi_j\phi_k - \sqrt{2}\varepsilon^{ijk}m_i\phi_i\bar{\phi}_j\bar{\phi}_k - \frac{1}{2}m_i\psi_i^2 - \frac{1}{2}\bar{m}_i\bar{\psi}_i^2\right] \end{aligned} \tag{5.10}$$

which gives the mass deformation $\mathcal{L}_m^{\mathcal{N}=4}$.

If we turn on $m_0, \bar{m}_0$, the Lagrangian contains the new terms

$$\delta\mathcal{L} = \mathrm{Tr}\left[\frac{\sqrt{2}}{3}\varepsilon^{ijk}m_0(\phi_i\phi_j\phi_k) - \frac{\sqrt{2}}{3}\varepsilon^{ijk}\bar{m}_0(\bar{\phi}_i\bar{\phi}_j\bar{\phi}_k) - \frac{1}{2}m_0\lambda^2 - \frac{1}{2}\bar{m}_0\bar{\lambda}^2\right]. \tag{5.11}$$

The first two terms are written as

$$-\frac{1}{kg_{\mathrm{YM}}^2}\frac{\sqrt{2}}{3}\varepsilon^{ijk}\int d^2\theta\ \mathrm{Tr}\left[e^{m_0\theta^2}\Phi_i\Phi_j\Phi_k\right] + \frac{1}{kg_{\mathrm{YM}}^2}\frac{\sqrt{2}}{3}\varepsilon^{ijk}\int d^2\bar{\theta}\ \mathrm{Tr}\left[e^{\bar{m}_0\bar{\theta}^2}\bar{\Phi}_i\bar{\Phi}_j\bar{\Phi}_k\right], \tag{5.12}$$

which are regarded as the deformation of superpotential. The last two terms are regarded as deformation of gauge coupling constant:

$$\frac{1}{16kg_{\mathrm{YM}}^2}\mathrm{Tr}\left[\int d^2\theta\ e^{2m_0\theta^2}W^2 + \int d^2\bar{\theta}\ e^{2\bar{m}_0\bar{\theta}^2}\bar{W}^2\right]. \tag{5.13}$$

The Lagrangian (5.5) together with (5.12) and (5.13) becomes the (A,S)-type deformed one.

[2] Here we assume the weight factor $\frac{2}{3}$ for the amplitude $\langle\langle V_{H_{\varphi\varphi}}V_\varphi V_{\mathcal{F}}\rangle\rangle$, which could be determined by evaluating the five-point amplitude $\langle\langle V_\varphi V_\varphi V_\varphi V_{\mathcal{F}}\rangle\rangle$.

5.2 $\mathcal{N}=2$ deformation

We can also write down the deformed $\mathcal{N}=2$ Lagrangians in $\mathcal{N}=1$ superspace. The $\mathcal{N}=2$ super Yang-Mills theory in $\mathcal{N}=1$ superspace is given by

$$\mathcal{L}^{\mathcal{N}=2} = \frac{1}{kg_{\rm YM}^2}\int d^2\theta d^2\bar{\theta}\ \mathrm{Tr}\left[\bar{\Phi}e^{2V}\Phi e^{-2V}\right] + \frac{1}{16kg_{\rm YM}^2}\mathrm{Tr}\left[\int d^2\theta\ W^\alpha W_\alpha + \int d^2\bar{\theta}\ \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right], \tag{5.14}$$

where Φ and $\bar{\Phi}$ are chiral and anti-chiral superfields.

For the (S,A)-type deformation, the interaction terms (4.5) are written as

$$\mathcal{L}^{(1)}_{\rm (S,A)} + \mathcal{L}^{(2)}_{\rm (S,A)} = -\frac{1}{2kg_{\rm YM}^2}\int d^4\theta\ \theta^2\bar{\theta}^2\mathrm{Tr}\left[\bar{\Phi}D_\alpha W_\beta \tilde{C}^{\alpha\beta}\right] + \frac{1}{2kg_{\rm YM}^2}\int d^2\bar{\theta}\ \bar{\theta}^2\left[\bar{\Phi}^2\tilde{C}^{\mu\nu}\tilde{C}_{\mu\nu}\right]. \tag{5.15}$$

Therefore the (S,A)-type deformation in superspace is realized by introducing new interaction term in the D- and F-terms. It would be interesting to examine this deformation in terms of $\mathcal{N}=2$ superspace and its relation to the Ω-deformation of $\mathcal{N}=2$ super Yang-Mills theory [16, 34].

We now discuss the (A,S)-type deformation. As in the $\mathcal{N}=4$ case, the (A,S)-type deformation is realized by the deformation of coupling parameters. The mass deformation of $\mathcal{N}=2$ super Yang-Mills theory is described by the Lagrangian

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=2}_m &= \mathcal{L}^{\mathcal{N}=2} + \frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[\frac{1}{2}\int d^2\theta\ m\Phi^2 + \frac{1}{2}\int d^2\bar{\theta}\ \bar{m}\bar{\Phi}^2\right]\\ &= \mathcal{L}^{\mathcal{N}=2} + \mathrm{Tr}\left[-|m|^2|\phi|^2 - \frac{1}{2}m\psi^2 - \frac{1}{2}\bar{m}\bar{\psi}^2\right].\end{aligned} \tag{5.16}$$

Here we have integrated out auxiliary fields of the superfields. If we diagonalize the background $C^{(ij)} = \mathrm{diag}(\bar{m}_0, \bar{m})$, the (A,S)-type deformation term (4.13) is written as

$$\mathcal{L}^{(1)}_{\rm (A,S)} = \frac{1}{kg_{\rm YM}^2}\mathrm{Tr}\left[-\bar{m}_0\bar{\lambda}^2 - \bar{m}\bar{\psi}^2\right]. \tag{5.17}$$

The second term gives mass deformation term with $m=0$. On the other hand, the first term is written in terms of superspace valued gauge coupling as

$$\frac{1}{16kg_{\rm YM}^2}\int d^2\bar{\theta}\left[e^{4\bar{m}_0\bar{\theta}^2}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right]. \tag{5.18}$$

6 Conclusions and Discussion

In this paper we studied the first and second order corrections from the constant R-R 3-form backgrounds to $\mathcal{N}=2$ and 4 super Yang-Mills theories, which are realized as the low-energy effective field theories on the (fractional) D3-branes in type IIB superstring theory. We argued the (S,A) and (A,S)-type R-R backgrounds $\mathcal{F}$, which correspond to the R-R (dual) 3-form field strengths in closed superstring backgrounds. We also used the scaling condition, where $(\alpha')^{1/2}\mathcal{F}$ is fixed in the zero-slope limit, to calculate the disk amplitudes including a closed string R-R vertex operator.

The (S,A)-type background with this scaling condition is particularly useful to study non-perturbative effects of super Yang-Mills theory. In fact, the instanton effective action of $\mathcal{N}=2$ super Yang-Mills theory with the (S,A)-type deformation agrees with that in Ω-background at the lowest order in the deformation parameter and gauge coupling constant [15]. The Ω-background is an important setup to applying a localization formula to the integration over the instanton moduli space [3, 16, 33, 34]. It is an interesting problem to extend this correspondence to $\mathcal{N}=4$ theory or the (A,S)-type deformation and examine how the instanton moduli space and the low energy-effective action are deformed by this background since the (S,A) and (A,S) deformed $\mathcal{N}=4$ super Yang-Mills theories can accommodate both self-dual tensor and vector backgrounds simultaneously from the viewpoint of $\mathcal{N}=2$ deformations. This is also important in order to study the nonperturbative superstring vacua in the presence of R-R backgrounds.

We examined supersymmetry of the deformed $\mathcal{N}=4$ action and find that $\mathcal{N}=4$ supersymmetry is broken for generic deformation parameter. But for special case, a part of supersymmetries are unbroken and also deformed by the R-R background, which are similar to the non(anti)commutative superspace. We argued the rank condition for deformation parameter to determine unbroken supersymmetries. Deformations of $\mathcal{N}=2$ and $\mathcal{N}=1$ super Yang-Mills theories are described by the orbifold construction, and the number of unbroken supersymmetries are determined by the rank condition for deformation parameters. It would be interesting to study how the central charge of extended supersymmetry algebra is deformed in these backgrounds.

The (S,A) and (A,S)-type deformations of action cannot be realized by the defor-

mation in non(anti)commutative superspace [10] since the spinor index structure of the background is different from that of non(anti)commutative superspace. In the scaling condition $(\alpha')^{1/2}\mathcal{F}$ fixed, we find that the (A,S)-type deformation of $\mathcal{N}=2$ super Yang-Mills theory is realized by allowing coupling constants to take values in $\mathcal{N}=1$ superspace. For the (S,A) case, we need to introduce further interaction terms for superfields. But some interaction terms take simple form when we use $\mathcal{N}=2$ extended superspace. Therefore it would be interesting to examine this deformation as the geometry of $\mathcal{N}=2$ superspace, as discussed in the case of Ω-background deformation [33, 34].

Acknowledgments

The work of K. I. is supported in part by the Grant-in-Aid for Scientific Research No. 18540255 from Ministry of Education, Science, Culture and Sports of Japan. The work of H. N. is supported by the Postdoctoral Research Program of Sungkyunkwan University (2007) and is the result of research activities (Astrophysical Research Center for the Structure and Evolution of the Cosmos (ARCSEC)) supported by KOSEF. The work of S. S. is supported by bilateral exchange program between Japan Society for the Promotion of Science (JSPS) and the Academy of Finland.

References

[1] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. **B413** (1994) 162, hep-th/9307158.

[2] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. **B405** (1993) 279, hep-th/9302103; Commun. Math. Phys. **165** (1994) 311, hep-th/9309140.

[3] N. A. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831, hep-th/0206161.

[4] R. Dijkgraaf and C. Vafa, Nucl. Phys. **B644** (2002) 3, hep-th/0206255.

[5] C. S. Chu and P. M. Ho, Nucl. Phys. **B550** (1999) 151, hep-th/9812219.

[6] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999), hep-th/9908142.

[7] H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. **7** (2003) 53, hep-th/0302109; Adv. Theor. Math. Phys. **7** (2004) 405, hep-th/0303063.

[8] N. Berkovits and N. Seiberg, JHEP **0307** (2003) 010, hep-th/0306226.

[9] J. de Boer, P. A. Grassi and P. van Nieuwenhuizen, Phys. Lett. **B574** (2003) 98, hep-th/0302078.

[10] N. Seiberg, JHEP **0306** (2003) 010, hep-th/0305248.

[11] T. Araki, K. Ito and A. Ohtsuka, Phys. Lett. **B573** (2003) 209, hep-th/0307076.

[12] S. Ferrara, E. Ivanov, O. Lechtenfeld, E. Sokatchev and B. Zupnik, Nucl. Phys. **B704** (2005) 154, hep-th/0405049.

[13] A. De Castro, E. Ivanov, O. Lechtenfeld, L. Quevedo, Nucl. Phys. **B747** (2006) 1, hep-th/0510013;
A. De Castro and L. Quevedo, Phys. Lett. **B639** (2006) 117, hep-th/0605187.

[14] C. Saemann and M. Wolf, JHEP **0403** (2004) 048, hep-th/0401147.

[15] M. Billó, M. Frau, F. Fucito and A. Lerda, JHEP **0611** (2006) 012, hep-th/0606013.

[16] A. S. Losev, A. Marshakov and N. A. Nekrasov, hep-th/0302191.

[17] M. Billó, M. Frau, I. Pesando and A. Lerda, JHEP **0405** (2004) 023, hep-th/0402160.

[18] K. Ito and S. Sasaki, JHEP **0611** (2006) 004, hep-th/0608143.

[19] K. Ito, Y. Kobayashi and S. Sasaki, JHEP **0704** (2007) 011, hep-th/0612267.

[20] T. Araki and K. Ito, Phys. Lett. B **595** (2004) 513 hep-th/0404250.

[21] R. C. Myers, JHEP **9912** (1999) 022, hep-th/9910053.

[22] A. Imaanpur, JHEP **0503** (2005) 030, hep-th/0501167;
R. Abbaspur and A. Imaanpur, JHEP **0601** (2006) 017, hep-th/0509220.

[23] D. Friedan, E. J. Martinec and S. H. Shenker, Nucl. Phys. **B271** (1986) 93.

[24] V. A. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, Nucl. Phys. **B288** (1987) 173.

[25] M. Dine, I. Ichinose and N. Seiberg, Nucl. Phys. **B293** (1987) 253;
J. J. Atick, L. J. Dixon and A. Sen Nucl. Phys. **B292** (1987) 109.

[26] M. Billó, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, JHEP **0302** (2003) 045, hep-th/0211250.

[27] T. Araki, K. Ito and A. Ohtsuka, JHEP **0401** (2004) 046, hep-th/0401012.

[28] T. Araki, K. Ito and A. Ohtsuka, Phys. Lett. B **606** (2005) 202, hep-th/0410203.

[29] K. Ito and H. Nakajima, Phys. Lett. B **629** (2005) 93, arXiv:hep-th/0508052.

[30] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, Nucl. Phys. **B621** (2002) 157, hep-th/0107057;
M. Bertolini, P. Di Vecchia and R. Marotta, hep-th/0112195.

[31] M. Bertolini, P. Di Vecchia, G. Ferretti and R. Marotta, Nucl. Phys. **B630** (2002) 222, hep-th/0112187.

[32] J. Wess and J. Bagger, “Supersymmetry and Supergravity”, Princeton University Press, 1992.

[33] N. A. Nekrasov and A. Okounkov, hep-th/0306238.

[34] S. Schadchin, hep-th/0502180.